BOUNDS FOR TOEPLITZ DETERMINANTS OF A CERTAIN SET OF ANALYTIC FUNCTIONS DEFINED BY q-DIFFERENTIATION AND HYPERBOLIC COSINE FUNCTION

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Abstract

In complex analysis, determinants are used to study properties of matrices arising from complex-valued functions, particularly in transformations, residue computations, Jacobians for conformal mappings, systems of complex equations, and in evaluating special matrices like Toeplitz and Hankel. Thus, this study is on a certain set $M_q(\alpha, \beta, t)$ consisting of analytic and univalent functions of the Taylor's series $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$ defined by using and maps the unit disk $\mathbb{U}$ onto a domain defined by the hyperbolic cosine function $t(z) = \cosh\sqrt{z}$. In the methodology, the principles of quantum derivative (q-derivative) operator, subordination, q-series expansion, and some widely acknowledged lemmas are adopted. The established results include the initial coefficient bounds for $|a_n|$ $(n = \{2,3,4,5\})$ and Toeplitz determinants: $|\mathbb{T}_{2,1}(f)|, |\mathbb{T}_{2,2}(f)|, |\mathbb{T}_{2,3}(f)|,$ and $|\mathbb{T}_{3,1}(f)|$. These results however generalised many existing ones thereby expanding the scope of its applications in areas such as in the solution to analytical problems of orthogonal polynomials, determinants, and frequency analysis where the special cases of the Toeplitz determinants are involved.

Keywords: Analytic function, univalent function, hyperbolic cosine function, Toeplitz determinant, coefficient bounds.

MSC: 30C45, 30C50.

1. INTRODUCTION

Complex functions, which involve variables and outputs in the complex plane, form a cornerstone of modern mathematical analysis. They extend real-valued functions into the realm of complex numbers, revealing richer structures, analyses, and properties, such as analyticity, functionals, and conformality. Within this framework, the hyperbolic cosine function

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

arises naturally as an analytic function of a complex variable z. It shares similarities with the trigonometric cosine function but exhibits distinct geometric and analytic behaviours, particularly in mapping and transformation properties of points, lines, curves, and shapes in the

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complex plane. Its analytic nature allows it to be represented by power series expansions, making it a useful tool in solving differential equations, modeling wave phenomena, and describing shapes in hyperbolic geometry.

Toeplitz matrices, on the other hand, originate in linear algebra but connect deeply with complex function theory through their generating functions, often expressed in terms of Taylor's or Laurent's series. When the generating function of a Toeplitz matrix involves analytic functions like the hyperbolic cosine function, the matrix inherits structural patterns and spectral properties from the function's analytic behaviour. This relationship is particularly important in areas such as signal processing, numerical analysis, and the study of integral equations, where Toeplitz determinants and their asymptotics are linked to special functions, such as $\cosh z$. Thus, the interplay between complex functions, the hyperbolic cosine, and Toeplitz matrices provides a bridge between abstract function theory and practical computational applications.

In this work, we study bounds for some Toeplitz determinants in connection with a certain set of complex functions that are mapped into the domain of a certain hyperbolic cosine function.

2. LITERATURE REVIEW
2.1 Some Analytic Functions and Definitions

Complex functions are mathematical mappings (with transformation functions) that assign to each complex number $z = x + iy$ a complex value. They extend the idea of real-valued functions into the complex plane, thereby allowing richer behaviour and deeper structural properties. When a complex function

$$f(z = x + iy) = u(x, y) + iv(x, y), \quad x, y \in \mathbb{R}$$

is analytic, then it must surely satisfies the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and is infinitely differentiable within a chosen domain. Of interest in this study is the set of analytic and univalent functions of the power series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n \qquad (2.1)$$

where $a_1 = 1$ and a_n are complex coefficients determined by the function's derivatives at a point. This representation is not only elegant but also central to the study of coefficient problems in complex analysis. Coefficient problems concern the determination, estimation, and characterization of the coefficients a_n for functions within specific classes, such as bounded turning functions, starlike functions, convex functions, and several others. These problems are important in geometric function theory, where bounds on coefficients provide information about the mapping properties of function f, such as its growth, distortion, and area coverage. A notable historical example is the Bieberbach conjecture, which proposed precise bounds on the coefficients of normalized univalent functions and was proved by Louis de Branges in 1985.

Other coefficient problems arise in the study of subclasses defined by (q)-differential operators, (q)-integral operators, and functional inequalities, often leading to extremal functions that achieve these bounds.

The applications of complex functions and coefficient theory extend far beyond pure mathematics. In engineering, analytic functions are used to solve two-dimensional potential problems in electrostatics, fluid dynamics, and heat conduction through conformal mappings that preserve angles and local shapes. In physics, they appear in wave propagation, quantum mechanics, and relativity theory. In signal processing, analytic functions are related to Fourier transforms and filter design, where coefficient bounds can help ensure system stability and desired frequency characteristics. In number theory, they underpin the study of generating functions, modular forms, and zeta functions, connecting analysis to arithmetic properties.

Therefore, the study of complex functions, their coefficients, and associated extremal problems forms a bridge between abstract mathematical theory and practical applications. The interplay between analytic structure and coefficient constraints not only enriches our understanding of complex function behaviour but also provides powerful tools for solving real-world scientific and engineering problems.

Going forward, let $\mathbb{N}$ and $\mathbb{C}$ represent the usual sets of natural and complex numbers, respectively. Also, let

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

represent the unit disk and let $\mathbb{A}$ represent the set of complex-valued functions defined in $\mathbb{U}$. More so, let $\mathbb{S}$ represent the set of those functions that are both analytic and univalent in $\mathbb{U}$. It is well-known that functions in $\mathbb{S}$ can by definition be expressed in Taylor's series form (1.1) and normalized by the conditions $f(0) = 0 = f'(0) - 1$, and $z \in \mathbb{U}$. Furthermore, represented by " $\prec$ " is the notation for subordination. Thus, we say that for two functions $f_1, f_2 \in \mathbb{A}$, $f_1 \prec f_2$ if and only if we get a Schwarz function

$$\xi(z) = \varepsilon_1 z + \varepsilon_2 z^2 + \varepsilon_3 z^3 + \cdots$$

normalized by the conditions $\xi(0) = 0, |\xi(z)| = |z| < 1$, and $z \in \mathbb{U}$ so that the conditions
$$f_1(z) = f_2(z) \circ \xi(z) = f_2(\xi(z))$$

are satisfied. Now, should $f_2(z)$ be a univalent function, then $f_1(z) \prec f_2(z)$ if and only if the conditions

$$f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{U}) \subset f_2(\mathbb{U})$$
are satisfied.

Another set of functions observed in this work is the set $\mathbb{H}$ of Carathéodory functions. This set is known to consists of analytic functions whose real parts are positive in $\mathbb{U}$. In general, functions in set $\mathbb{H}$ can be represented in Taylor's series form

$$P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \qquad (2.2)$$

and normalized by the conditions $\Re e P(z) > 0$, $P(0) = 1$, and $z \in \mathbb{U}$. It is well-known that functions $P(z)$ and $\xi(z)$ have an interrelationship defined by

$$\xi(z) = \frac{P(z) - 1}{P(z) + 1} = \frac{(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots) - 1}{(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots) + 1}$$

which simplifies to

$$\xi(z) = \frac{1}{2} p_1 z + \left(\frac{1}{2} p_2 - \frac{1}{4} p_1^2 \right) z^2 + \left(\frac{1}{8} p_1^3 - \frac{1}{2} p_1 p_2 + \frac{1}{2} p_3 \right) z^3 + \cdots. \qquad (2.3)$$

2.2 Hyperbolic Cosine Functions

The hyperbolic cosine function, denoted by $\cosh z$, is defined for a complex variable z by

$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

It is an even analytic function whose series expansion is expressed as

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

and it is easy to see that it converges everywhere in the complex plane, $\mathbb{C}$. Closely related to the trigonometric cosine, $\cosh z$ differs in that it arises naturally from hyperbolic geometry and models growth rather than oscillation. In the context of complex analysis, $\cosh z$ inherits all the properties of analytic functions, including differentiability and integrability at every point, and representation through power series or contour integrals.

The connection between $\cosh z$ and complex functions lies in its analytic structure, which makes it a member of the broader family of exponential-type functions. Through Euler's formula for complex arguments, $\cosh z$ is linked to the cosine function via the identities $\cosh(iz) = \cos z$ and $\cos(iz) = \cosh z$. This duality law allows results from trigonometric function theory to be translated into the hyperbolic setting, and vice versa. Moreover, $\cosh z$ frequently appears in the study of entire functions, Fourier transforms, and in solving partial differential equations with complex variables.

Applications of the hyperbolic cosine span mathematics, physics, and engineering. In physics, it describes the shape of a hanging cable (the catenary) and appears in solutions to the one-dimensional wave and heat equations. In special relativity, $\cosh$ functions model Lorentz transformations, relating time and space coordinates between inertial frames. In engineering, they are used in stability analysis, signal processing, and modeling exponential-like growth

phenomena. Thus, $\cosh z$ serves as both a fundamental object in complex function theory and a versatile tool in applied sciences.

In this work, we studied some geometric properties of the hyperbolic cosine complex function

$$t(z) = \cos\sqrt{z}\,. \tag{2.4}$$

Observe that function $t(z)$ is analytic in the unit disk $\mathbb{U}$, thus, we can apply the subordination principle by substituting (1.3) into (1.4) to get

$$t(\xi(z)) = \cos\sqrt{\xi(z)} = 1 + \frac{1}{4}p_1 z + \frac{1}{4}\left(p_2 - \frac{11}{24}p_1^2\right)z^2 + \frac{1}{4}\left(\frac{301}{1440}p_1^3 - \frac{11}{24}p_1 p_2 + p_3\right)z^3$$
$$+ \frac{1}{4}\left(-\frac{91}{960}p_1^4 + \frac{301}{480}p_1^2 - \frac{11}{12}p_1 p_3 - \frac{11}{24}p_2^2 + p_4\right)z^4 + \cdots. \tag{2.5}$$

This function will be observed later.

2.3 Toeplitz Determinants

Toeplitz determinants arise from Toeplitz matrices, which are structured matrices whose entries remain constant along each diagonal from left to right. A Toeplitz determinant is simply the determinant of such a matrix, and its value depends on the sequence $\{T_n\}$ that defines the diagonals. These determinants have deep connections with functional analysis, probability, and mathematical physics due to their role in describing systems with translational or time-invariant properties.

The connection between Toeplitz determinants and complex functions appears through their generating functions, often called symbols. If a function $f(re^{i\theta})$, $z = e^{i\theta}$, defined on the unit circle is integrable, then its Fourier coefficients $\hat{f}_n$ can be used as the entries of a Toeplitz matrix. In this way, the determinant of the Toeplitz matrix encodes analytic information about f. If f is analytic inside the unit disk, then some powerful results from complex analysis, such as contour integration and factorization theorems can be applied to study the asymptotic behaviour of the determinants as the matrix size grows. This link is particularly important in Szegö's limit theorem, which relates the asymptotics of large Toeplitz determinants to the exponential of an integral involving $\log f$. Special functions, including the hyperbolic cosine, can serve as generating functions, tying Toeplitz determinants directly to problems in analytic function theory.

Applications of Toeplitz determinants span multiple disciplines. In statistical mechanics, they appear in the computation of partition functions, especially in models like the two-dimensional Ising model. In random matrix theory, they help describe eigenvalue correlations and limiting distributions. In signal processing and time-series analysis, Toeplitz matrices naturally model autocorrelation structures, making their determinants useful in spectral estimation and filter design. In numerical analysis, their structured form allows for efficient algorithms in solving large linear systems of equations and in image reconstruction. Furthermore, in operator theory, Toeplitz determinants are connected to spectral properties of Toeplitz operators, which have applications in control theory and complex dynamical systems.

Thus, Toeplitz determinants serve as a meeting point between linear algebra, complex function theory, and applied mathematics. Their ability to encode analytic properties of generating functions gives them both theoretical richness and practical utility, bridging abstract mathematical structures with concrete problems in science and engineering.

By considering the function f in (2.1), Thomas and Halim (2017) introduced the *symmetric Toeplitz determinant* defined by

$$\mathbb{T}_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & a_{n+3} & \cdots & a_{n+m-2} \\ a_{n+2} & a_{n+3} & a_n & \cdots & a_{n+m-3} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m-2} & a_{n+m-3} & \cdots & a_n \end{vmatrix}$$

where $m, n \in \mathbb{N}$, $a_1 = 1$, and a_n are the coefficients of f in (2.1). A careful selection of values for m and n shows that

$$\left.\begin{aligned} |\mathbb{T}_{2,1}(f)| &= |1 - a_2^2| \\ |\mathbb{T}_{2,2}(f)| &= |a_2^2 - a_3^2| \\ |\mathbb{T}_{2,3}(f)| &= |a_3^2 - a_4^2| \\ |\mathbb{T}_{3,1}(f)| &= |1 - 2a_2^2 - a_3(a_3 - 2a_2^2)| \end{aligned}\right\} \qquad (2.6)$$

Interested researchers may check Thomas and Halim (2017) and Ramachandran and Kavitha (2017) for more details.

3. METHODOLOGY
3.1 *q*-Derivative Operator

The q-derivative of a function $h(x)$, usually denoted by $\mathfrak{D}_q h(x)$, was initiated by Jackson (1910) where it is a generalization of the classical derivative defined by

$$\mathfrak{D}_q h(x) = \frac{h(qx) - h(x)}{x(q-1)} \quad (x \neq 0, q \neq 1).$$

Nonetheless, the q-derivative reduces to the classical derivative as $q \to 1$. In complex analysis, q-derivative act on analytic functions, preserving many of their structural properties while introducing a discrete scaling aspect. This makes them central to q-calculus (q-differentiation, q-integration, q-series, q-analysis, and so on), which extends differential and integral calculus to settings involving a deformation of parameter q.

The connection with complex functions arises because q-derivatives can be applied to entire univalent, bi-univalent, harmonic, and meromorphic functions, producing q-analogues of classical results like Taylor and Laurent series. Applications include solving q-difference equations in mathematical physics, studying special functions such as basic hypergeometric series, modeling quantum groups in representation theory, and analyzing discrete versions of wave and heat equations in engineering and applied sciences.
For a complex function f of the form (2.1), the q-derivative is defined by

$$\left.\begin{aligned} \mathcal{D}_q f(0) &= f'(0) = 1 \\ \mathcal{D}_q f(z) &= \frac{f(qz) - f(z)}{z(q-1)} \ (z \neq 0) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \\ \mathcal{D}_q^2 f(z) &= D_q \left(D_q f(z) \right) = \sum_{n=2}^{\infty} [n-1]_q [n]_q a_n z^{n-2} \end{aligned}\right\} \qquad (3.1)$$

where

$q \in (0,1)$, $[n]_q = \frac{1-q^n}{1-q}$, $\lim_{q \uparrow 1} [n]_q = n$ implies $\lim_{q \uparrow 1} \mathcal{D}_q f(z) = \lim_{q \uparrow 1} \frac{f(qz)-f(z)}{z(q-1)} = f'(z)$.

For example,

1. $f_1(z) = z \Longrightarrow \mathcal{D}_q f(z) = [1]_q = 1$ and $\mathcal{D}_q^2 f(z) = 0$.
2. $f_2(z) = z + cz^3 \Longrightarrow \mathcal{D}_q f(z) = 1 + [3]_q cz^2$ and $\mathcal{D}_q^2 f(z) = [2]_q [3]_q cz$.

More details on q-derivative can be found in Lasode et al. (2025).

3.2   Relevant Lemmas

The following lemmas hold and shall be applied in the course of proving the theorems.

Lemma 3.1. If $P(z)$ is as given in (2.2), then $|p_n| \leq 2$ for every $n \in \mathbb{N}$ (Thomas et al., 2018).

Lemma 3.2. If $P(z)$ is as given in (2.2), then
$|p_{m+n} - X p_m p_n| \leq 2$ for $X \in [0,1]$
(Ravichandran and Verma, 2015).

Lemma 3.3. If $P(z)$ is as given in (2.2), then for $J, K, L \in \mathbb{R}$,
$|J p_1^3 - K p_1 p_2 + L p_3| \leq 2|J| + |-2J + K| + 2|J - K + L|$
(Arif et al., 2019).

Lemma 3.4. If $P(z)$ is as given in (2.2); $J, K, L, M \in \mathbb{R}$;
$0 < K < 1, 0 < L < 1$, and
$8K(1-K)\{(LM - 2J)^2 + [L(K+L) - M]^2 + L(1-L)(M - 2KL)\} \leq 4KL^2 (1-L)^2 (1-K)$,
then
$\left| J p_1^4 + K p_2^2 + 2L p_1 p_3 - \frac{3}{2} M p_1^2 p_{2-p_4} \right| \leq 2$
(Ravichandran and Verma, 2015).

Lemma 3.5. If $P(z)$ is as given in (2.2), then for $U \in \mathbb{C}$,
$\left| p_2 - U \frac{p_1^2}{2} \right| \leq 2\max\{1, |1 - U|\}$
(Babalola and Opoola, 2008).

4.   MAIN RESULTS
4.1   A New Set of Analytic Functions

Definition 4.1. Function f of the infinite series form (2.1) is said to be a member of the set $M_q(\alpha, \beta, t)$ if the subordination condition

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$$1 + \frac{1}{\alpha}\{\mathfrak{D}_q f(z) + \beta z \mathfrak{D}_q^2 f(z) - 1\} \prec t(z) \qquad (4.1)$$

is satisfied for the conditions: $\alpha, \beta \in \mathbb{C}$, $q \in (0,1)$, $t(z) = \cosh\sqrt{z}$, and $\mathfrak{D}_q f(z)$ is as defined in (3.1). Thus, to avoid repetitions, we shall let the declarations of the parameters $\alpha, \beta \in \mathbb{C}$, $q \in (0,1)$, $(z) = \cosh\sqrt{z}$, and $\mathfrak{D}_q f(z)$ hold throughout this work.

4.2 Initial Coefficients Bounds for $f \in M_q(\alpha, \beta, t)$

Theorem 4.1 If f in (2.1) is in the set $M_q(\alpha, \beta, t)$, then

$$|a_2| \leqq \frac{|\alpha|}{2[2]_q|1 + \beta|}, \qquad (4.2)$$

$$|a_3| \leqq \frac{|\alpha|}{2[3]_q|1 + [2]_q\beta|}, \qquad (4.3)$$

$$|a_4| \leqq \frac{|\alpha|}{2[4]_q|1 + [3]_q\beta|}, \qquad (4.4)$$

$$|a_5| \leqq \frac{|\alpha|}{2[5]_q|1 + [4]_q\beta|}, \qquad (4.5)$$

and in general,

$$|a_n| \leqq \frac{|\alpha|}{2[n]_q|1 + [n-1]_q\beta|} \qquad (n \geqq 2).$$

Proof. If $f \in M_q(\alpha, \beta, t)$, then by the subordination principle, (4.1) can be expressed as

$$1 + \frac{1}{\alpha}\{D_q f(z) + \beta z D_q^2 f(z) - 1\} = t(\xi(z)). \qquad (4.6)$$

Observe that from (2.1) and (3.1) we will get

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots,$$
$$D_q f(z) = 1 + [2]_q a_2 z + [3]_q a_3 z^2 + [4]_q a_4 z^3 + [5]_q a_5 z^4 + \cdots, \qquad (4.7)$$

and

$$\beta z D_q^2 f(z) = [1]_q[2]_q \beta a_2 z + [2]_q[3]_q \beta a_3 z^2 + [3]_q[4]_q \beta a_4 z^3 + [4]_q[5]_q \beta a_5 z^4 + \cdots \qquad (4.8)$$

where putting (4.7) and (4.8) into LHS of (4.6) gives

$$1 + \frac{1}{\alpha}\{D_q f(z) + \beta z D_q^2 f(z) - 1\} = 1 + \frac{(1 + \beta)[2]_q}{\alpha} a_2 z + \frac{(1 + [2]_q\beta)[3]_q}{\alpha} a_3 z^2$$

$$+ \frac{(1 + [3]_q\beta)[4]_q}{\alpha} a_4 z^3 + \frac{(1 + [4]_q\beta)[5]_q}{\alpha} a_5 z^4 + \cdots. \qquad (4.9)$$

Recall that the RHS of (4.6) was earlier given by (2.5), hence, comparing the coefficients of (4.9) and (2.5) shows that

$$\frac{(1 + \beta)[2]_q}{\alpha} a_2 = \frac{1}{4} p_1 \Longrightarrow a_2 = \frac{\alpha}{4[2]_q(1 + \beta)} p_1 \qquad (4.10)$$

and

$$|a_2| = \left| \frac{\alpha}{4[2]_q(1+\beta)} p_1 \right| \Rightarrow |a_2| \leqq \frac{|\alpha|}{4[2]_q|1+\beta|} |p_1|.$$

Hence, the application of Lemma 2.1 shows that

$$|a_2| \leqq \frac{|\alpha|}{4[2]_q|1+\beta|} \quad (2)$$

and some simplifications give (4.2). Secondly,

$$\frac{(1+[2]_q\beta)[3]_q}{\alpha} a_3 = \frac{1}{4}\left(p_2 - \frac{11}{24}p_1^2\right) \Rightarrow a_3 = \frac{\alpha}{4[3]_q(1+[2]_q\beta)}\left(p_2 - \frac{11}{24}p_1^2\right) \quad (4.11)$$

and

$$|a_3| = \left| \frac{\alpha}{4[3]_q(1+[2]_q\beta)}\left(p_2 - \frac{11}{24}p_1^2\right) \right| \Rightarrow |a_3| \leqq \frac{|\alpha|}{4[3]_q|1+[2]_q\beta|}\left| p_2 - \frac{11}{24}p_1^2 \right|.$$

Hence, the application of Lemma 2.2 (where $X = \frac{11}{24}$) shows that

$$|a_3| \leqq \frac{|\alpha|}{4[3]_q|1+[2]_q\beta|} \quad (2)$$

and some simplifications give (4.3). Thirdly,

$$\frac{(1+[3]_q\beta)[4]_q}{\alpha} a_4 = \frac{1}{4}\left(\frac{301}{1440}p_1^3 - \frac{11}{12}p_1 p_2 + p_3\right)$$

implies

$$a_4 = \frac{\alpha}{4[4]_q(1+[3]_q\beta)}\left(\frac{301}{1440}p_1^3 - \frac{11}{12}p_1 p_2 + p_3\right) \quad (4.12)$$

and

$$|a_4| \leqq \frac{|\alpha|}{4[4]_q|1+[3]_q\beta|}\left| \frac{301}{1440}p_1^3 - \frac{11}{12}p_1 p_2 + p_3 \right|.$$

Hence, the application of Lemma 2.3 (where $= \frac{301}{1440}, K = \frac{11}{12}, L = 1$) shows that

$$|a_4| \leqq \frac{|\alpha|}{4[4]_q|1+[3]_q\beta|} \quad (2)$$

and some simplifications give (4.4). And lastly,

$$\frac{(1+[4]_q\beta)[5]_q}{\alpha} a_5 = \frac{1}{4}\left(-\frac{91}{960}p_1^4 + \frac{11}{24}p_2^2 - \frac{11}{12}p_1 p_3 + \frac{301}{480}p_1^2 p_2 + p_4\right)$$

implies

$$a_5 = \frac{\alpha}{4[5]_q(1+[4]_q\beta)}\left(-\frac{91}{960}p_1^4 + \frac{11}{24}p_2^2 - \frac{11}{12}p_1 p_3 + \frac{301}{480}p_1^2 p_2 + p_4\right)$$

and

$$|a_5| \leq \frac{|\alpha|}{4[5]_q|1 + [4]_q\beta|}\left|-\frac{91}{960}p_1^4 + \frac{11}{24}p_2^2 - \frac{11}{12}p_1p_3 + \frac{301}{480}p_1^2p_2 + p_4\right|.$$

Hence, the application of Lemma 2.4 (where $= \frac{91}{960}$, $K = \frac{11}{24}$, $L = \frac{11}{12}$, $M = \frac{301}{480}$) shows that

$$|a_5| \leq \frac{|\alpha|}{4[5]_q|1 + [4]_q\beta|}(2).$$

and some simplifications give (4.5).

4.3 Bounds for some Toeplitz Determinants

Theorem 4.2 If f in (2.1) is in the set $M_q(\alpha, \beta, t)$, then

$$|\mathbb{T}_{2,1}(f)| \leq 1 + \frac{|\alpha|}{2[2]_q|1 + \beta|}.$$

Proof. Observe that from (2.6),

$$|\mathbb{T}_{2,1}(f)| = |1 - a_2^2| \leq 1 + |a_2^2|$$

and using (4.2) implies that

$$|\mathbb{T}_{2,1}(f)| \leq 1 + \frac{|\alpha|}{2[2]_q|1 + \beta|}$$

which complete the proof.

Theorem 4.3 If f in (2.1) is in the set $M_q(\alpha, \beta, t)$, then

$$|\mathbb{T}_{2,2}(f)| \leq \frac{|\alpha|^2}{4}\left\{\frac{1}{[2]_q^2|1 + \beta|^2} + \frac{1}{[3]_q^2|1 + [2]_q\beta|^2}\right\}.$$

Proof. Observe that putting (4.10) and (4.11) into (2.6) means

$$\mathbb{T}_{2,2}(f) = a_2^2 - a_3^2 = \left[\frac{\alpha}{4[2]_q(1 + \beta)}p_1\right]^2 - \left[\frac{\alpha}{4[3]_q(1 + [2]_q\beta)}\left(p_2 - \frac{11}{24}p_1^2\right)\right]^2$$

$$= \frac{\alpha^2}{16[2]_q^2(1 + \beta)^2}p_1^2 - \frac{\alpha^2}{16[3]_q^2(1 + [2]_q\beta)^2}\left(p_2 - \frac{11}{24}p_1^2\right)^2$$

so that

$$|\mathbb{T}_{2,2}(f)| \leq \frac{|\alpha|^2}{16[2]_q^2|1 + \beta|^2}|p_1^2| + \frac{|\alpha|^2}{16[3]_q^2|1 + [2]_q\beta|^2}\left|p_2 - \frac{11}{24}p_1^2\right|^2.$$

Hence, the application of Lemmas 3.1 and 3.2 shows that

$$|\mathbb{T}_{2,2}(f)| \leq \frac{|\alpha|^2}{16[2]_q^2|1 + \beta|^2}(2)^2 + \frac{|\alpha|^2}{16[3]_q^2|1 + [2]_q\beta|^2}(2)^2$$

and some simplifications complete the proof.

Theorem 4.4 If f in (2.1) is in the set $M_q(\alpha, \beta, t)$, then

$$|\mathbb{T}_{2,3}(f)| \leqq \frac{|\alpha|^2}{4}\left\{\frac{1}{[3]_q^2|1+[2]_q\beta|^2} + \frac{1}{[4]_q^2|1+[3]_q\beta|^2}\right\}.$$

Proof. Observe that putting (4.11) and (4.12) into (2.6) means

$$\mathbb{T}_{2,3}(f) = a_3^2 - a_4^2$$

$$= \left[\frac{\alpha}{4[3]_q(1+[2]_q\beta)}\left(p_2 - \frac{11}{24}p_1^2\right)\right]^2 - \left[\frac{\alpha}{4[4]_q(1+[3]_q\beta)}\left(\frac{301}{1440}p_1^3 - \frac{11}{12}p_1p_2 + p_3\right)\right]^2$$

$$= \frac{\alpha^2}{16[3]_q^2(1+[2]_q\beta)^2}\left(p_2 - \frac{11}{24}p_1^2\right)^2 - \frac{\alpha^2}{16[4]_q^2(1+[3]_q\beta)^2}\left(\frac{301}{1440}p_1^3 - \frac{11}{12}p_1p_2 + p_3\right)^2$$

so that

$$|\mathbb{T}_{2,3}(f)| \leqq \frac{\alpha^2}{16[3]_q^2(1+[2]_q\beta)^2}\left|p_2 - \frac{11}{24}p_1^2\right|^2$$

$$+ \frac{\alpha^2}{16[4]_q^2(1+[3]_q\beta)^2}\left|\frac{301}{1440}p_1^3 - \frac{11}{12}p_1p_2 + p_3\right|^2.$$

Hence, the application of Lemmas 3.2 and 3.3 shows that

$$|\mathbb{T}_{2,3}(f)| \leqq \frac{\alpha^2}{16[3]_q^2(1+[2]_q\beta)^2}(2)^2 - \frac{\alpha^2}{16[4]_q^2(1+[3]_q\beta)^2}(2)^2$$

and some simplifications complete the proof.

Theorem 4.5 If f in (2.1) is in the set $M_q(\alpha, \beta, t)$, then

$$|\mathbb{T}_{3,1}(f)| \leqq 1 + \frac{|\alpha|^2}{2}\left[\frac{1}{[2]_q^2|1+\beta|^2} + \frac{1}{2[3]_q^2|1+[2]_q\beta|^2}\max\{1, V_q(\alpha, \beta)\}\right]$$

where

$$V_q(\alpha, \beta) = \left|\frac{1}{12} - \frac{\alpha[3]_q(1+[3]_q\beta)}{[2]_q^2(1+\beta)^2}\right|.$$

Proof. Observe that from (2.6),

$$|\mathbb{T}_{3,1}(f)| = |1 - 2a_2^2 - a_3(a_3 - 2a_2^2)| \leqq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|. \qquad (4.13)$$

Foremost and using (4.10) and (4.11) means

$$a_3 - 2a_2^2 = \frac{\alpha}{4[3]_q(1+[2]_q\beta)}\left(p_2 - \frac{11}{24}p_1^2\right) - 2\left[\frac{\alpha p_1}{4[2]_q(1+\beta)}\right]^2$$

$$= \frac{\alpha}{4[3]_q(1+[2]_q\beta)}p_2 - \frac{11\alpha}{96[3]_q(1+[2]_q\beta)}p_1^2 - \frac{\alpha^2}{8[2]_q^2(1+\beta)^2}p_1^2$$

$$= \frac{\alpha}{4[3]_q(1+[2]_q\beta)}p_2 - \frac{\alpha}{8}\left[\frac{11}{12[3]_q(1+[2]_q\beta)} + \frac{\alpha}{[2]_q^2(1+\beta)^2}\right]p_1^2$$

$$= \frac{\alpha}{4[3]_q(1+[2]_q\beta)}\left\{p_2 - \left[\frac{11}{12} + \frac{\alpha[3]_q(1+[2]_q\beta)}{[2]_q^2(1+\beta)^2}\right]\frac{p_1^2}{2}\right\}$$

so that

$$|a_3 - 2a_2^2| \leqq \frac{|\alpha|}{4[3]_q|1+[2]_q\beta|}\left|p_2 - \left(\frac{11}{12} + \frac{\alpha[3]_q(1+[2]_q\beta)}{[2]_q^2(1+\beta)^2}\right)\frac{p_1^2}{2}\right|$$

or for brief,

$$|a_3 - 2a_2^2| \leqq \frac{|\alpha|}{4[3]_q|1+[2]_q\beta|}\left|p_2 - U_q(\alpha,\beta)\frac{p_1^2}{2}\right| \qquad (4.14)$$

where

$$U_q(\alpha,\beta) = \frac{11}{12} + \frac{\alpha[3]_q(1+[2]_q\beta)}{[2]_q^2(1+\beta)^2}.$$

Application of Lemma 3.5 implies that

$$\left|1 - U_q(\alpha,\beta)\right| = \left|1 - \left(\frac{11}{12} + \frac{\alpha[3]_q(1+[2]_q\beta)}{[2]_q^2(1+\beta)^2}\right)\right| = \left|\frac{1}{12} - \frac{\alpha[3]_q(1+[3]_q\beta)}{[2]_q^2(1+\beta)^2}\right| = V_q(\alpha,\beta)$$

so that (4.14) becomes

$$|a_3 - 2a_2^2| \leqq \frac{|\alpha|}{4[3]_q|1+[2]_q\beta|}2\max\{1, V_q(\alpha,\beta)\}$$

$$= \frac{|\alpha|}{2[3]_q|1+[3]_q\beta|}\max\{1, V_q(\alpha,\beta)\}. \qquad (4.15)$$

Now, using (4.2), (4.3) and (4.15) in (4.13) yields

$$|\mathbb{T}_{3,1}(f)| \leqq 1 + 2\left(\frac{|\alpha|}{2[2]_q|1+\beta|}\right)^2$$

$$+ \left(\frac{|\alpha|}{2[3]_q|1+[2]_q\beta|}\right)\left(\frac{|\alpha|}{2[3]_q|1+[2]_q\beta|}\max\{1, V_q(\alpha,\beta)\}\right)$$

$$= 1 + \frac{2|\alpha|^2}{4[2]_q^2|1+\beta|^2} + \frac{|\alpha|^2}{4[3]_q^2|1+[2]_q\beta|^2}\max\{1, V_q(\alpha,\beta)\}$$

and some simplifications complete the proof.

5.   DISCUSSION

The importance and applications of calculus in almost all ramification of human endeavours cannot be underrated. In particular, q-calculus which is the q-analogous version of the classical calculus was adopted in this study, where the q-differentiation was used to define the new set $M_q(\alpha, \beta)$, therefore generalising many known classical differential operators. As a result of this classic provision from q-differentiation, the properties of the new set of analytic and univalent functions, defined in the unit disk was studied within the space of a certain hyperbolic cosine function for the bounds for some Toeplitz determinants. These properties involve four special cases of Toeplitz determinants: $|\mathbb{T}_{2,1}(f)|$, $|\mathbb{T}_{2,2}(f)|$, $|\mathbb{T}_{2,3}(f)|$, and $|\mathbb{T}_{3,1}(f)|$, whose bounds exist as evident in the theorems. Nonetheless,

(i) the new set generalises many existing and new ones when the involving parameters are varied;

(ii) the results improve some existing ones;

(iii) the results unify some existing ones;

(iv) the results open new directions of research in geometric function theory, where complex functions bridge the gap between Toeplitz determinants and a hyperbolic cosine function;

(v) the results therefore, broaden and open new areas of applications in conformal mapping which has applications in fluid mechanics, aerodynamics and in some areas of engineering designs and constructions.

6.   RECOMMENDATION

The study recommends that the results from work be applied in solving various physical problems in engineering designs and constructions, heat flows, aerodynamics, hydrodynamics, fluid flows and in the solution to analytical problems of polynomials, determinants and frequency analysis where the special cases of the Toeplitz determinants are involved.

7.   CONCLUSION

In conclusion, we were able to define a new set $M_q(\alpha, \beta)$ of functions f that are analytic and univalent in the unit disk. The definition of the new set of functions involves q-differentiation, subordination, and hyperbolic cosine function. In Theorem 4.1, we established bounds for the coefficients of functions $f \in M_q(\alpha, \beta)$, the bounds were generalized for $n \in \mathbb{N}$. Furthermore, In Theorems 4.2 - 4.5, the coefficients of $f \in M_q(\alpha, \beta)$ formed the elements of a symmetric Toeplitz determinants whose bounds were investigated and established. The new set however generalised many existing ones and the results from this work extended many known and new ones when the underlying parameters are varied. Invariably, the new set and its results broaden the scope of applications of the existing ones in available literature and practical.

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